

# Ring-type singular solutions of the biharmonic nonlinear Schrödinger equation

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## Abstract

We present new singular solutions of the biharmonic nonlinear Schrödinger equation

$$i\psi_t(t, \mathbf{x}) - \Delta^2 \psi + |\psi|^{2\sigma} \psi = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad 4/d \leq \sigma \leq 4.$$

These solutions collapse with the quasi self-similar ring profile  $\psi_{Q_B}$ , where

$$|\psi_{Q_B}(t, r)| \sim \frac{1}{L^{2/\sigma}(t)} Q_B \left( \frac{r - r_{\max}(t)}{L(t)} \right), \quad r = |\mathbf{x}|,$$

$L(t)$  is the ring width that vanishes at singularity,  $r_{\max}(t) \sim r_0 L^\alpha(t)$  is the ring radius, and  $\alpha = \frac{4-\sigma}{\sigma(d-1)}$ . The blowup rate of these solutions is  $\frac{1}{3+\alpha}$  for  $4/d \leq \sigma < 4$ , and slightly faster than  $1/4$  for  $\sigma = 4$ . These solutions are analogous to the ring-type solutions of the nonlinear Schrödinger equation.

## 1 Introduction

The focusing nonlinear Schrödinger equation (NLS)

$$i\psi_t(t, \mathbf{x}) + \Delta \psi + |\psi|^{2\sigma} \psi = 0, \quad \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}) \in H^1(\mathbb{R}^d), \quad (1)$$

where  $\mathbf{x} \in \mathbb{R}^d$  and  $\Delta = \partial_{x_1 x_1} + \dots + \partial_{x_d x_d}$ , admits solutions that become singular at a finite time, i.e.,  $\lim_{t \rightarrow T_c} \|\psi\|_{H^1} = \infty$ , where  $0 \leq t \leq T_c$ . Until a few years ago, all known singular solutions of the NLS were peak-type. By this, we mean that if assume radial symmetry, and denote the location of maximal amplitude by

$$r_{\max}(t) = \arg \max_r |\psi|, \quad r = \sqrt{x_1^2 + \dots + x_d^2},$$

then  $r_{\max}(t) \equiv 0$  for  $0 \leq t \leq T_c$ , i.e., the solution peak is attained at  $r = 0$ . In recent years, however, new singular solutions of the NLS were found, which are ring-type, i.e.,  $r_{\max}(t) > 0$  for  $0 \leq t < T_c$ .

In this study, we consider the focusing *biharmonic* nonlinear Schrödinger equation (BNLS)

$$i\psi_t(t, \mathbf{x}) - \Delta^2 \psi + |\psi|^{2\sigma} \psi = 0, \quad \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}) \in H^2(\mathbb{R}^d), \quad (2)$$

where  $\Delta^2$  is the biharmonic operator. Singular peak-type solutions of the BNLS have been studied in [FIP02, BFM10, BF]. Singular ring-type solutions of the BNLS with  $\sigma > 4$  were studied in [BFG09]. The goal of this work is to find and characterize singular ring-type solutions of the BNLS with  $4/d \leq \sigma \leq 4$ .

### 1.1 Singular solutions of the nonlinear Schrödinger equation (NLS) - review

The NLS (1) is called *subcritical* if  $\sigma d < 2$ . In this case, all solutions exist globally. In contrast, solutions of the *critical* ( $\sigma d = 2$ ) and *supercritical* ( $\sigma d > 2$ ) NLS can become singular at a finite time.

Until a few years ago, the only known singular NLS solutions were peak-type. In the critical case  $\sigma d = 2$ , it has been rigorously shown [MR03] that peak-type solutions are self-similar near the singularity, i.e.,  $\psi \sim \psi_R$ , where

$$\psi_R(t, r) = \frac{1}{L^{d/2}(t)} R\left(\frac{r}{L(t)}\right) e^{i \int_0^t \frac{ds}{L^2(s)}},$$

and  $r = |\mathbf{x}|$ . The self-similar profile  $R(\rho)$  is the ground state of the standing-wave equation

$$R''(\rho) + \frac{d-1}{\rho} R' - R + |R|^{4/d} R = 0.$$

Since  $R$  attains its global maximum at  $\rho = 0$ ,  $\psi_R$  is a peak-type profile. The blowup rate of  $L(t)$  is given by the *loglog law*

$$L(t) \sim \left( \frac{2\pi(T_c - t)}{\log \log 1/(T_c - t)} \right)^{1/2}, \quad t \rightarrow T_c. \quad (3)$$

In the supercritical case  $\sigma d > 2$ , the rigorous theory is far less developed. However, formal calculations and numerical simulations [SS99] suggest that peak-type solutions of the supercritical NLS collapse with the self-similar  $\psi_S$  profile, i.e.,  $\psi \sim \psi_S$ , where

$$\psi_S(t, r) = \frac{1}{L^{1/\sigma}(t)} S(\rho) e^{i\tau}, \quad (4a)$$

$$\tau = \int_0^t \frac{ds}{L^2(s)}, \quad \rho = \frac{r}{L(t)}, \quad (4b)$$

and  $S(\rho)$  is the zero-Hamiltonian, monotonically-decreasing solution of the nonlinear eigenvalue problem

$$S''(\rho) + \frac{d-1}{\rho} S' - S + i \frac{\kappa^2}{2} \left( \frac{1}{\sigma} S + \rho S' \right) + |S|^{2\sigma} S = 0, \quad S'(0) = 0, \quad (4c)$$

where  $\kappa$  is the eigenvalue. Since  $|S(\rho)|$  attains its global maximum at  $\rho = 0$ ,  $\psi_S$  is a peak-type profile. The blowup rate of  $L(t)$  is a square-root, i.e.,

$$L(t) \sim \kappa \sqrt{T_c - t}, \quad t \rightarrow T_c, \quad (4d)$$

where  $\kappa > 0$  is the eigenvalue of (4c).

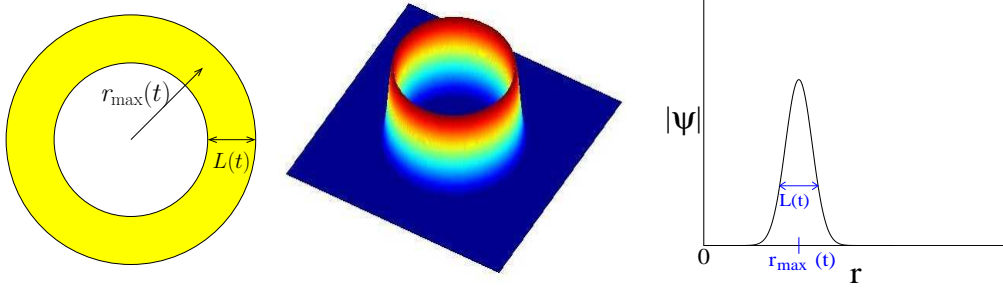
In the last few years, new singular solutions of the NLS were discovered, which are ring-type [FGW05, FGW07, Rap06, RS09, FG08]. In particular, in [FGW07], Fibich, Gavish and Wang showed that the NLS (1) with  $d > 1$  and  $\frac{2}{d} \leq \sigma \leq 2$  admits singular ring-type solutions that collapse with the  $\psi_Q$  profile, i.e.,  $\psi \sim \psi_Q$ , where

$$\psi_Q(t, r) = \frac{1}{L^{1/\sigma}(t)} Q(\rho) e^{i\tau + i\alpha_{\text{NLS}} \frac{L(t)}{4L} r^2 + i(1 - \alpha_{\text{NLS}}) \frac{L(t)}{4L} (r - r_{\max}(t))^2}, \quad (5a)$$

$$\tau = \int_0^t \frac{ds}{L^2(s)}, \quad \rho = \frac{r - r_{\max}(t)}{L(t)}, \quad r_{\max}(t) \sim r_0 L^{\alpha_{\text{NLS}}}(t), \quad (5b)$$

and

$$\alpha_{\text{NLS}} = \frac{2 - \sigma}{\sigma(d - 1)} = 1 - \frac{\sigma d - 2}{\sigma(d - 1)}. \quad (5c)$$

Figure 1: Illustration of ring radius  $r_{\max}(t)$  and width  $L(t)$ .

The self-similar profile  $Q$  attains its global maximum at  $\rho = 0$ . Hence,  $r_{\max}(t)$  is the ring radius and  $L(t)$  is the ring width, see Figure 1.

A unique feature of the  $\psi_Q$  profile (5a) is the linear combination of the two radial phase terms. The first phase term  $\alpha_{\text{NLS}} \frac{L_t}{4L} r^2$  describes focusing towards  $r = 0$ , and is the manifestation of the shrinking of the ring radius  $r_{\max}$  to zero. The second term  $(1 - \alpha_{\text{NLS}}) \frac{L_t}{4L} (r - r_{\max}(t))^2$  describes focusing towards  $r = r_{\max}$ , and is the manifestation of the shrinking of the ring width  $L(t)$  to zero. The discovery of this “double-lens” ansatz was the key stage in the asymptotic analysis of the  $\psi_Q$  profile, which enabled the calculation of the shrinking rate  $\alpha_{\text{NLS}}$ , see (5c) and the blowup rate  $p$ , see (7).

The NLS ring-type singular solutions can be classified as follows, see Figure 2:

- A. In the subcritical case ( $\sigma d < 2$ ), all NLS solutions exist globally, hence no singular ring-type solutions exist.
- B. The critical case  $\sigma d = 2$  corresponds to  $\alpha_{\text{NLS}} = 1$ . Since  $r_{\max}(t) \sim r_0 L(t)$ , these solutions undergo an *equal-rate collapse*, i.e., the ring radius goes to zero at the same rate as  $L(t)$ . The blowup rate of  $L(t)$  is a square root.
- C. The supercritical case  $2/d < \sigma < 2$  corresponds to  $0 < \alpha_{\text{NLS}} < 1$ . Therefore, the ring radius  $r_{\max}(t) \sim r_0 L^{\alpha_{\text{NLS}}}(t)$  decays to zero, but at a slower rate than  $L(t)$ . The blowup rate of  $L(t)$  is

$$L(t) \sim \kappa(T_c - t)^p, \quad (6)$$

where

$$p = \frac{1}{1 + \alpha_{\text{NLS}}} = \frac{1}{2 - \frac{\sigma d - 2}{\sigma(d-1)}} \quad (7)$$

- D. The supercritical case  $\sigma = 2$  corresponds to  $\alpha_{\text{NLS}} = 0$ , i.e.,  $\lim_{t \rightarrow T_c} r_{\max}(t) = r_{\max}(T_c) > 0$ . Therefore, the solution becomes singular on the  $d$ -dimensional sphere  $|\mathbf{x}| = r_{\max}(T_c)$ , rather than at a point. The blowup profile  $\psi_Q$  is equal to that of peak-type solutions of the 1D critical NLS, and the blowup rate is given by the loglog law (3).
- E. The case  $\sigma > 2$  also corresponds to a standing ring. The asymptotic profile is not given by  $\psi_Q$ , however, but rather by the asymptotic profile of peak-type solutions of the 1D supercritical NLS. The blowup rate is a square root [BFG09].

Thus, NLS ring-type singular solutions are *shrinking* (i.e.,  $\lim_{t \rightarrow T_c} r_{\max}(t) = 0$ ) for  $\frac{2}{d} \leq \sigma < 2$  (cases B and C), and *standing* (i.e.,  $0 < \lim_{t \rightarrow T_c} r_{\max}(t) < \infty$ ) for  $\sigma \geq 2$  (cases D and E).

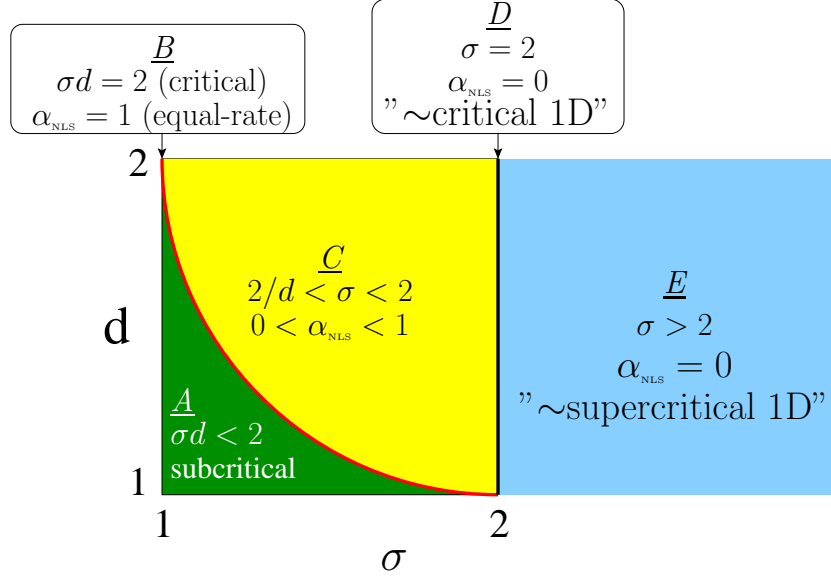


Figure 2: Classification of singular ring-type solutions of the NLS, as a function of  $\sigma$  and  $d$ . A) subcritical case - no singular solutions exist. B)  $\sigma d = 2$ : equal-rate  $\psi_Q$  solutions [FGW05]. C)  $2/d < \sigma < 2$ : shrinking  $\psi_Q$  solutions [FGW07]. D)  $\sigma = 2$ : standing  $\psi_Q$  solutions [FGW07, Rap06, RS09]. E)  $\sigma > 2$ : standing non- $\psi_Q$  rings [BFG09].

## 1.2 Singular solutions of the biharmonic NLS - review

The BNLS equation (2) is called *subcritical* if  $\sigma d < 4$ , *supercritical* if  $\sigma d > 4$ , and *critical* if  $\sigma d = 4$ . In the critical case, equation (2) can be rewritten as

$$i\psi_t(t, \mathbf{x}) - \Delta^2 \psi + |\psi|^{8/d} \psi = 0, \quad \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}) \in H^2(\mathbb{R}^d). \quad (8)$$

The BNLS conserves the “power” ( $L^2$  norm), i.e.,

$$P(t) \equiv P(0), \quad P(t) = \|\psi(t)\|_2^2,$$

and the Hamiltonian

$$H(t) \equiv H(0), \quad H[\psi(t)] = \|\Delta \psi\|_2^2 - \frac{1}{1+\sigma} \|\psi\|_{2(\sigma+1)}^{2(\sigma+1)}. \quad (9)$$

In the radially-symmetric case, the BNLS equation (2) reduces to

$$i\psi_t(t, r) - \Delta_r^2 \psi + |\psi|^{2\sigma} \psi = 0, \quad \psi(0, r) = \psi_0(r), \quad (10)$$

where

$$\Delta_r^2 = \partial_r^4 + \frac{2(d-1)}{r} \partial_r^3 + \frac{(d-1)(d-3)}{r^2} \partial_r^2 - \frac{(d-1)(d-3)}{r^3} \partial_r \quad (11)$$

is the radial biharmonic operator. In particular, the radially-symmetric critical BNLS is given by

$$i\psi_t(t, r) - \Delta_r^2 \psi + |\psi|^{8/d} \psi = 0, \quad \psi(0, r) = \psi_0(r). \quad (12)$$

All solutions of the subcritical BNLS exist globally in  $H^2$  [FIP02]. In the critical case, they exist globally if the input power is below the critical power:

**Theorem 1** ([FIP02]). *Let  $\|\psi_0\|_2^2 < P_{cr}^B$ , where  $P_{cr}^B = \|R_B\|_2^2$ , and  $R_B(\rho)$  is the ground state of the standing wave equation*

$$-\Delta_\rho^2 R_B(\rho) - R_B + |R_B|^{2\sigma} R_B = 0, \quad (13)$$

*with  $\sigma = 4/d$ . Then, the solution of the critical focusing BNLS (8) exists globally.*

Numerical simulations [FIP02, BFM10] indicate that solutions of the critical and supercritical BNLS can become singular at a finite time, i.e.,  $\lim_{t \rightarrow T_c} \|\psi\|_{H^2} = \infty$ , where  $0 < T_c < \infty$ . At present, however, there is no rigorous proof that the BNLS admits singular solutions, whether peak-type or ring-type.

In [BFM10], we rigorously proved that the blowup rate of all  $H^2$  singular solutions of the critical BNLS is bounded by a quartic root:

**Theorem 2.** *Let  $\psi$  be a solution of the critical BNLS (8) that becomes singular at  $t = T_c < \infty$ , and let  $l(t) = \|\Delta\psi\|_2^{-1/2}$ . Then,  $\exists K = K(\|\psi_0\|_2) > 0$  such that*

$$l(t) \leq K(T_c - t)^{1/4}, \quad 0 \leq t < T_c.$$

We also proved that all singular solutions are quasi self-similar:

**Theorem 3.** *Let  $d \geq 2$ , and let  $\psi(t, r)$  be a solution of the radially-symmetric critical BNLS (12) with initial conditions  $\psi_0(r) \in H_{\text{radial}}^2$ , that becomes singular at  $t = T_c < \infty$ . Let  $l(t) = \|\Delta\psi\|_2^{-1/2}$ , and let*

$$S(\psi)(t, r) = l^{d/2}(t)\psi(t, l(t)r).$$

*Then, for any sequence  $t'_k \rightarrow T_c$ , there is a subsequence  $t_k$ , such that  $S(\psi)(t_k, r) \rightarrow \Psi(r)$  strongly in  $L^q$ , for all  $q$  such that*

$$\begin{cases} 2 < q < \infty & 2 \leq d \leq 4, \\ 2 < q < \frac{2d}{d-4} & 4 < d. \end{cases} \quad (14)$$

Since the  $L^2$ -norm of  $S(\psi)$  is conserved, and the convergence of  $S(\psi)$  to  $\Psi$  is in  $L^q$  with  $q > 2$ , the solution becomes self-similar in the singular region (the collapsing core), but not everywhere. Consequentially, the solution has the power-concentration property, whereby a finite amount of power enters the singularity point, i.e.,

$$\lim_{\varepsilon \rightarrow 0+} \liminf_{t \rightarrow T_c} \|\psi\|_{L^2(r < \varepsilon)}^2 \geq P_{cr}^B,$$

where  $P_{cr}^B$  is the critical power for collapse [BFM10, CHK<sup>+</sup>].

Peak-type singular solutions of the critical BNLS (12) were studied asymptotically and numerically in [FIP02, BFM10]. The asymptotic profile of these solutions is

$$\psi_{R_B}(t, r) = \frac{1}{L^{d/2}(t)} R_B\left(\frac{r}{L(t)}\right) e^{i \int^t \frac{1}{L^4(s)} ds}, \quad (15)$$

where  $R_B(\rho)$  is the ground state of (13). The blowup rate of  $L(t)$  is slightly faster than a quartic root, i.e.,

$$\lim_{t \rightarrow T_c} \frac{L(t)}{(T_c - t)^p} = \begin{cases} 0 & p = \frac{1}{4} \\ \infty & p > \frac{1}{4} \end{cases} \quad (16)$$

Specifically, in the one-dimensional case, the quasi self-similar profile is

$$\psi_{R_{B,1D}}(t, x) = \frac{1}{L^{1/2}(t)} R_{B,1D}\left(\frac{x}{L(t)}\right) e^{i \int^t \frac{1}{L^4(s)} ds}, \quad (17a)$$

and  $R_{B,1D}$  is the ground state of

$$-R_B''''(\xi) - R_B + |R_B|^8 R_B = 0. \quad (17b)$$

Peak-type solutions of the supercritical BNLS (10) were studied asymptotically and numerically in [BF]. The asymptotic profile of these solutions is

$$\psi_{S_B}(t, r) = \frac{1}{L^{2/\sigma}(t)} S_B \left( \frac{r}{L(t)} \right) e^{i \int^t \frac{1}{L^4(s)} ds}, \quad (18)$$

where  $S_B(\rho)$  is the zero-Hamiltonian solution of a nonlinear eigenvalue problem

$$-S_B(\rho) + i \frac{\kappa^4}{4} \left( \frac{2}{\sigma} S_B + \rho S_B' \right) - \Delta_\rho S_B + |S_B|^{2\sigma} S_B = 0, \quad S_B'(0) = S_B'''(0) = 0, \quad (19)$$

and  $\kappa$  is the eigenvalue. The blowup rate is exactly  $p = 1/4$ , i.e.,

$$L(t) \sim \kappa (T_c - t)^{1/4}, \quad (20)$$

where  $\kappa > 0$  is the nonlinear eigenvalue of (19).

Ring-type singular solutions of the supercritical BNLS (10) with  $\sigma > 4$  were studied asymptotically and numerically in [BFG09]. These solutions are standing rings, i.e.,  $\lim_{t \rightarrow T_c} r_{\max}(t) > 0$ . The self-similar profile of these standing-ring solutions is

$$\psi_B(t, r) = \psi_{S_{B,1D}}(t, x = r - r_{\max}(t)) = \frac{1}{L^{2/\sigma}(t)} S_{B,1D} \left( \frac{r - r_{\max}(t)}{L(t)} \right) e^{i \int^t \frac{1}{L^4(s)} ds}, \quad (21)$$

where  $\psi_{S_{B,1D}}(t, x)$ , see (18), is the profile of the peak-type singular solution of the one-dimensional supercritical BNLS with the same value of  $\sigma$ . The blowup rate is given by (20).

### 1.3 Analogy of NLS and BNLS

Table ?? lists the major findings of the previous works [FIP02, BFM10, BFG09] on singular solutions of the BNLS, side by side with their NLS counterparts. In all cases, the results for the BNLS mirror those of the NLS exactly, “up to the change of  $2 \rightarrow 4$ ”.

We note that current BNLS theory is still missing a key feature in NLS theory, which is the BNLS analogue of the quadratic radial phase terms of the asymptotic profiles. Therefore, our asymptotic analysis of the BNLS singular solutions produces weaker results than those of [FGW07]. Hence, in this work we “fill in” the missing results by relying on the above analogy of the NLS and BNLS, up to the change  $2 \rightarrow 4$ .

### 1.4 Summary of results

In this study, we consider ring-type singular solutions of the BNLS (10) with  $4/d \leq \sigma \leq 4$ . We show numerically that such solutions exist, and are of the form  $\psi(t, r) \sim \psi_{Q_B}(t, r)$ , where

$$|\psi_{Q_B}(t, r)| = \frac{1}{L^{2/\sigma}(t)} Q_B(\rho), \quad (22a)$$

$$\rho = \frac{r - r_{\max}(t)}{L(t)}, \quad r_{\max}(t) \sim r_0 L^\alpha(t), \quad (22b)$$

and

$$\alpha = \alpha_B = \frac{4 - \sigma}{\sigma(d - 1)} = 1 - \frac{\sigma d - 4}{\sigma(d - 1)}. \quad (22c)$$

	NLS	BNLS
peak-type solutions: critical case [BFM10]		
	$\sigma d = 2$	$\sigma d = 4$
asymptotic profile	$\frac{1}{L^{1/\sigma}(t)} R\left(\frac{r}{L(t)}\right) e^{i \int_0^t \frac{ds}{L^2(s)}}$	$\frac{1}{L^{2/\sigma}(t)} R_B\left(\frac{r}{L(t)}\right) e^{i \int_0^t \frac{ds}{L^4(s)}}$
blowup rate	slightly faster than 1/2	slightly faster than 1/4
power concentration	yes	yes
peak-type solutions: supercritical case [BF]		
	$\sigma d > 2$	$\sigma d > 4$
asymptotic profile	$\frac{1}{L^{1/\sigma}(t)} S\left(\frac{r}{L(t)}\right) e^{i \int_0^t \frac{ds}{L^2(s)}}$	$\frac{1}{L^{2/\sigma}(t)} S_B\left(\frac{r}{L(t)}\right) e^{i \int_0^t \frac{ds}{L^4(s)}}$
blowup rate	$= 1/2$	$= 1/4$
power concentration	no	no
“supercritical” standing-ring solutions [BFG09]		
	$\sigma > 2$	$\sigma > 4$
asymptotic profile	same as 1D peak	same as 1D peak
blowup rate	same as 1D peak	same as 1D peak

Table 1: A comparison of the properties of singular solutions of the NLS and BNLS. These properties are analogous, “up to the change  $2 \rightarrow 4$ ”.

The  $\psi_{Q_B}$  profile is the BNLS analogue of the  $\psi_Q$  profile of the NLS. Unlike the  $\psi_Q$  profile, however, we do not know the expression for the double-lens ansatz of  $\psi_{Q_B}$ .

In Section 2, we consider the case  $\sigma = 4$ . In this case  $\alpha = 0$ , i.e., the solution is a singular standing ring. Informal asymptotic analysis and numerical simulations show that the blowup profile is the self-similar profile

$$\psi_{Q_B}(t, r; \sigma = 4) = \frac{1}{L^{1/2}(t)} R_{B,1D} \left( \frac{r - r_{\max}(t)}{L(t)} \right) e^{i \int_0^t \frac{1}{L^4(s)} ds}, \quad (23)$$

where  $R_{B,1D}$  is the ground state of (12) with  $\sigma = 4$  and  $d = 1$ , and that the blowup rate is slightly faster than a quartic root. In other words, the blowup rate and profile are the same as those of peak-type singular solutions of the one-dimensional critical BNLS, see (18).

In Section 3, we consider the case  $4/d < \sigma < 4$ , for which  $0 < \alpha_B < 1$ , see (22c). From power conservation we deduce that  $\alpha \geq \alpha_B$ . By analogy with the NLS, we expect that  $\alpha = \alpha_B$ . Therefore, the ring radius  $r_{\max}(t) \sim r_0 L^\alpha(t)$  decays to zero, but at a slower rate than  $L(t)$ . By analogy with the NLS, we also expect that the blowup rate of these ring solutions is given by (6) with

$$p = \frac{1}{4 - \frac{\sigma d - 4}{\sigma(d-1)}} = \frac{1}{3 + \alpha_B}.$$

Numerical experiments support these predictions.

In Section 4, we consider the critical BNLS ( $\sigma = 4/d$ ), which corresponds to  $\alpha_B = 1$ . Since the singular part of the solution has to be self-similar in  $r/L$ , see Theorem 3,  $\alpha$  must be equal to unity. By the analogy with the NLS, the blowup rate is conjectured to be  $1/4$ .

In summary, the BNLS singular ring-type solutions can be classified as follows (see Figure 3):

- A. In the subcritical case ( $\sigma d < 4$ ), all BNLS solutions exist globally, hence no collapsing ring solutions exist.
- B. The critical case  $\sigma d = 4$  corresponds to  $\alpha_B = 1$  (equal-rate collapse). The blowup rate is  $p = 1/4$ .

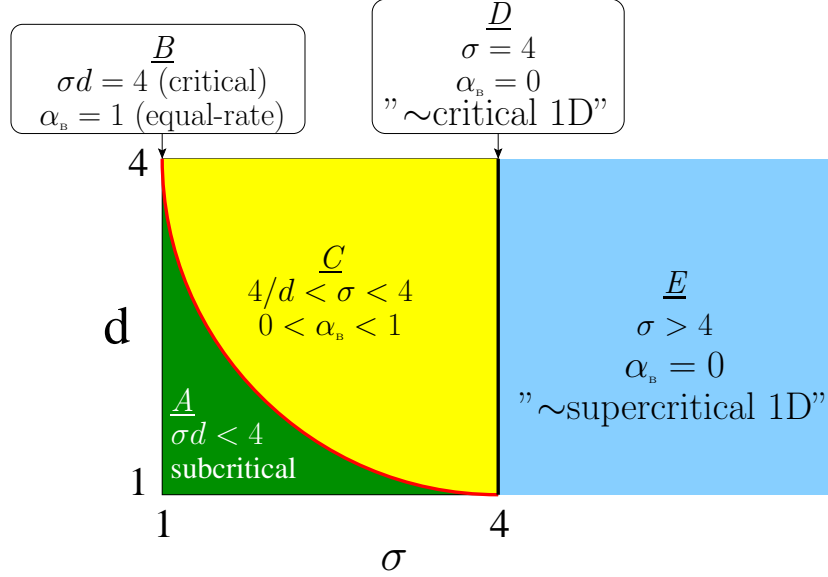


Figure 3: Classification of singular ring-type solutions of the BNLS as a function of  $\sigma$  and  $d$ . A: subcritical case (no singularity). B: critical case, with equal-rate collapse. C:  $4/d < \sigma < 4$ , shrinking rings. D:  $\sigma = 4$ , standing rings. E:  $\sigma > 4$ , standing rings [BFG09].

- C. The supercritical case  $4/d < \sigma < 4$  corresponds to  $0 < \alpha_B < 1$ , hence the ring radius  $r_{\max}(t)$  decays to zero, but at a slower rate than  $L(t)$ . The blowup rate is  $p = \frac{1}{4 - \frac{\sigma d - 4}{\sigma(d-1)}}$ .
- D. The case  $\sigma = 4$  corresponds to  $\alpha_B = 0$ . Hence the solution is a singular standing ring. The self-similar profile  $Q_B$  is equal to that of peak-type solutions of the 1D critical BNLS, and the blowup rate is slightly above  $p = 1/4$ .
- E. The case  $\sigma > 4$  was studied in [BFG09]. In this case, the solutions are of the standing-ring type, the self-similar profile is equal to that of peak-type solutions of the 1D supercritical BNLS, and the blowup-rate is a quartic-root.

Thus, up to the change  $2 \rightarrow 4$ , this classification is, indeed, completely analogous to that of singular ring-type solutions of the NLS (see Figure 2).

## 1.5 Numerical Methodology

The computations of singular BNLS solutions that focus by factors of  $10^8$  necessitated the usage of adaptive grids. For our simulations we developed a modified version of the Static Grid Redistribution method [RW00, DG09], which is much easier to implement in the biharmonic problem, and is easily extended to other evolution equations, such as the nonlinear heat and biharmonic nonlinear heat equations [BFG09]. The method of [DG09] also includes a mechanism for the prevention of under-resolution in the non-singular region. We extend this mechanism to prevent under-resolution in the transition layer between the singular and non-singular regions. See Section 5 for further details.



## 1.6 Critical exponents of singular ring solutions

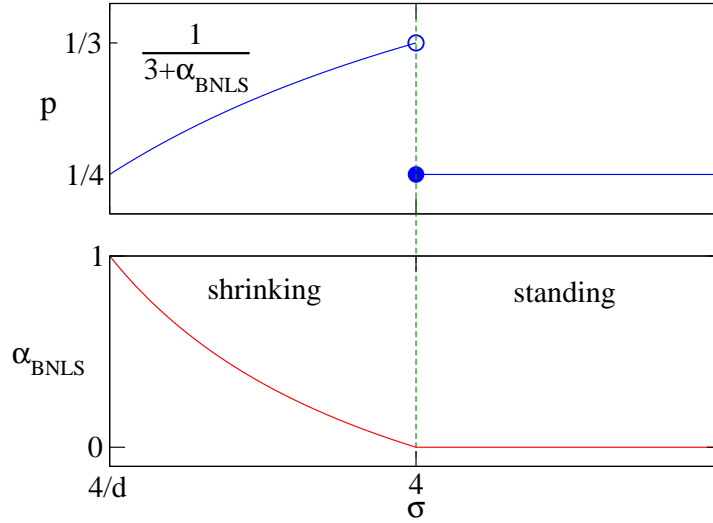


Figure 4: top: Blowup rate of singular ring solutions of the BNLS. The blowup rate increases monotonically from  $p = 1/4$  at  $\sigma = 4/d$  to  $p = (1/3)-$  at  $\sigma = 4-$ . For  $\sigma = 4$  (full circle)  $p = 1/4$  (with a loglog correction?) and for  $\sigma > 4$ ,  $p \equiv 1/4$ . bottom: The shrinkage parameter  $\alpha_{\text{B}}$  of singular ring solutions of the BNLS. For  $4/d \leq \sigma < 4$ ,  $\alpha_{\text{B}}$  decreases monotonically from 1 to  $0+$  (shrinking rings). For  $\sigma \geq 4$ ,  $\alpha_{\text{B}} \equiv 0$  (standing rings).

In Figure 4 (top) we plot the blowup rate  $p$  of singular ring solutions of the BNLS, see (6). As  $\sigma$  increases from  $4/d$  to  $4-$ ,  $p$  increases monotonically from  $\frac{1}{4}$  to  $\frac{1}{3}-$ . At  $\sigma = 4$ , the blowup rate is slightly faster than a quartic root, i.e.,  $p \approx \frac{1}{4}$ . Finally,  $p = 1/4$  for  $\sigma > 4$ . Since

$$\lim_{\sigma \rightarrow 4-} p = 1/3, \quad \lim_{\sigma \rightarrow 4+} p = \frac{1}{4},$$

the blowup rate has a discontinuity at  $\sigma = 4$ .

The above results show that  $\sigma = 4$  is a *critical exponent of singular ring solutions of the BNLS*. Intuitively, this is because the blowup dynamics changes from a shrinking-ring ( $\sigma < 4$ ) to a standing-ring ( $\sigma \geq 4$ ), see Figure 4 (bottom). We can understand why  $\sigma = 4$  is a critical exponent using the following argument. Standing-ring solutions are ‘equivalent’ to singular peak solutions of the one-dimensional NLS with the same nonlinearity exponent  $\sigma$  [BFG09]. Since  $\sigma = 4$  is the critical exponent for singularity formation in the one-dimensional NLS, it is also the critical exponent for standing-ring blowup. An analogous picture exists for the NLS, wherein the phase transition between standing and shrinking rings occurs at  $\sigma = 2$  [BFG09].

## 2 Singular standing rings ( $\sigma = 4$ )

In what follows, we show that collapse of ring-type singular solutions of the BNLS with  $\sigma = 4$  is “the same” as collapse of peak-type singular solutions of the one-dimensional critical BNLS.

### 2.1 Informal Analysis

We consider ring-type singular solutions of the supercritical BNLS (10) with  $\sigma = 4$ , that undergo a quasi self-similar collapse with the asymptotic profile

$$\psi_{Q_B}(t, r) = \frac{1}{L^{1/2}(t)} Q_B(\rho) e^{i \int^t \frac{1}{L^4(s)} ds} \quad \rho = \frac{r - r_{\max}(t)}{L}. \quad (24)$$

Here and throughout this paper, by quasi self-similar we mean that  $\psi \sim \psi_{Q_B}$  in the singular ring region  $r - r_{\max} = \mathcal{O}(L)$ , or  $\rho = \mathcal{O}(1)$ , but not for  $0 \leq r < \infty$ .

The asymptotic profile (24) describes a standing ring if  $\lim_{t \rightarrow T_c} r_{\max}(t) > 0$ . We expect ring-type singular solutions of the BNLS with  $\sigma = 4$  to collapse as standing rings, for the following two reasons:

1. By continuity, since ring-type singular solutions of the BNLS with  $\sigma > 4$  are standing rings [BFG09].
2. By analogy with singular ring-type solutions of the NLS with  $\sigma = 2$ , which are standing rings [FGW07, Rap06, RS09].

In the ring region  $r - r_{\max} = \mathcal{O}(L)$ , as  $L \rightarrow 0$ , the terms of the radial biharmonic operator (11) behave as

$$\left[ \frac{1}{r^{4-k}} \partial_r^k \psi \right] \sim \frac{[\psi]}{L^k}, \quad k = 0, \dots, 4.$$

Therefore,  $\Delta_r^2 \psi \sim \partial_r^4 \psi$ . Hence, near the singularity, equation (10) reduces to

$$\psi(t, r) - \psi_{rrrr} + |\psi|^8 \psi = 0,$$

which is the one-dimensional critical BNLS. Therefore, the singular solutions of the two equations are asymptotically equivalent, i.e.,

$$\psi_{\sigma=4,d}^{ring}(t, r) \sim \psi_{\sigma=4,d=1}^{peak}(t, x = r - r_{\max}(t)),$$

where  $\psi_{\sigma=4,d=1}^{peak}$  is a peak-type solution of the one-dimensional critical BNLS.

The above informal analysis suggests that the blowup dynamics of singular standing-ring solutions of the BNLS (10) with  $d > 1$  and  $\sigma = 4$  is the same as the blowup dynamics of singular peak solutions of the one-dimensional critical BNLS:

**Conjecture 4.** *Let  $d > 1$  and  $\sigma = 4$ , and let  $\psi$  be a singular ring-type solution of the BNLS (10). Then,*

1. *The solution is a standing ring, i.e.,  $\lim_{t \rightarrow T_c} r_{\max}(t) > 0$ .*
2. *In the ring region, the solution approaches the  $\psi_{Q_B}$  self-similar profile, see (24).*
3. *The self-similar profile  $\psi_{Q_B}$  is given by*

$$\psi_{Q_B}(t, r) = \psi_{R_{B,1D}}(t, x = r - r_{\max}(t)), \quad (25)$$

*where  $\psi_{R_{B,1D}}(t, x)$ , see (17), is the asymptotic profile of the one-dimensional critical BNLS.*

4. *Specifically,  $Q_B(\rho) = R_{B,1D}(\xi)$ , where  $R_{B,1D}(\xi)$  is the ground state of (17b).*
5. *The blowup rate of  $L(t)$  is slightly faster than a quartic root, see (16).*

In Section 2.2 we provide numerical evidence in support of Conjecture 4.

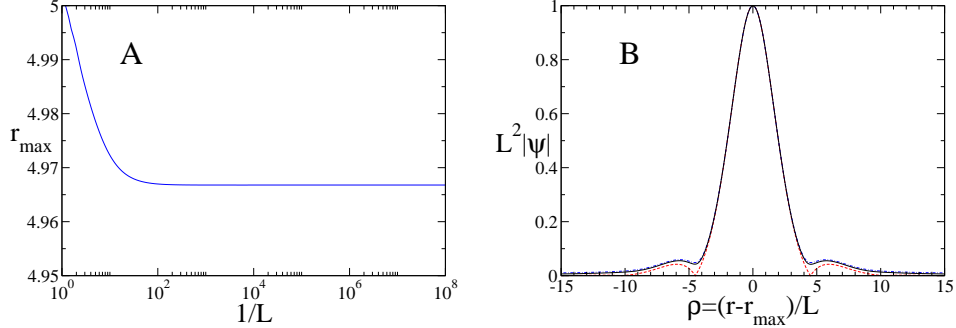


Figure 5: A singular standing-ring solution of the supercritical BNLS (10) with  $d = 2$  and  $\sigma = 4$ . A) Ring radius  $r_{\max}$  as a function of the focusing level  $1/L$ . B) The rescaled solution, see (26), at  $L(t) = 10^{-4}$  (blue dash-dotted line) and  $L(t) = 10^{-8}$  (black solid line). The two curves are indistinguishable. Red dashed line is the rescaled one-dimensional ground state  $|R_{B,1D}(x)|$ .

## 2.2 Simulations

The radially-symmetric BNLS (10) with  $d = 2$  and  $\sigma = 4$  was solved with the initial condition  $\psi_0(r) = 2e^{-(r-5)^2}$ . The simulation was run up to  $L = \mathcal{O}(10^{-8})$ . Similar results were obtained with  $d = 3$  and  $\sigma = 4$  (data not shown).

We next test each item of Conjecture 4 numerically:

1. The position of maximal amplitude  $r_{\max}(t) = \arg \max_r |\psi|$  approaches a positive constant as  $L \rightarrow 0$ , see Figure 5A, indicating that the solution collapses as a standing ring.
2. The solution profiles, at the focusing levels of  $L = 10^{-4}$  and  $L = 10^{-8}$ , rescaled according to

$$\psi_{\text{rescaled}}(t, \rho) = L^{2/\sigma}(t)\psi(t, r_{\max}(t) + \rho \cdot L), \quad L(t) = \|\psi\|_{\infty}^{-\sigma/2}, \quad (26)$$

are almost indistinguishable, see Figure 5B, indicating that the collapsing core is self-similar according to (24).

3. Figure 5B also shows that the self-similar profile of the standing-ring solution is given by  $R_{B,1D}(\xi)$ , the one-dimensional ground-state of equation (17b).
4. To calculate the blowup rate of  $\psi$ , we first assume that  $L(t) \sim \kappa(T_c - t)^p$ , and find the best fitting  $\kappa$  and  $p$ , see Figure 6A. In this case  $p \approx 0.2523$ , indicating that the blowup rate is a quartic-root or slightly faster.
5. In order to check whether the blowup rate of  $L$  is slightly faster than a quartic-root, we compute the limit  $\lim_{t \rightarrow T_c} L^3 L_t$ . Recall that for a quartic-root blowup rate  $L(t) \sim \kappa(T_c - t)^{1/4}$  with  $\kappa > 0$ ,

$$\lim_{t \rightarrow T_c} L^3 L_t = -\frac{\kappa^4}{4} < 0,$$

while for a faster-than-a quartic-root blowup rate, see (16),  $L^3 L_t$  goes to zero. Figure 6B shows that  $L^3 L_t$  does not approach a negative constant, but increases slowly towards  $0^-$ , implying that the blowup rate is slightly faster than a quartic root.

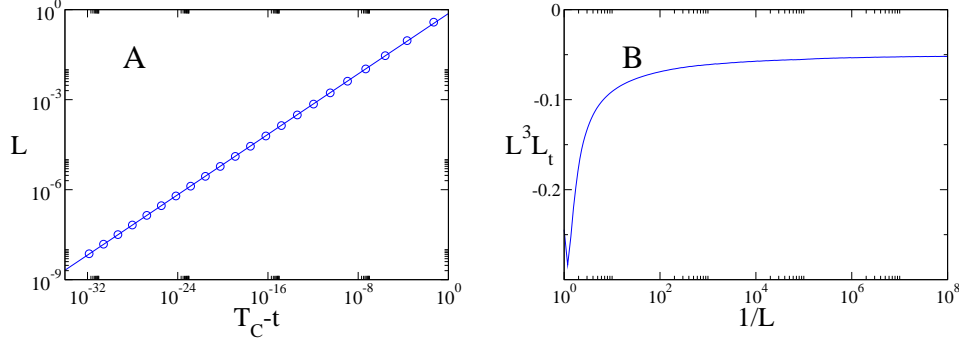


Figure 6: Blowup rate of the solution of Figure 5. A)  $L$  as a function of  $(T_c - t)$  on a logarithmic scale (circles). Solid line is  $L = 0.774(T_c - t)^{0.2523}$ . B)  $L_t L^3$  as a function of  $1/L$ .

Note that the initial condition  $\psi_0 = 2e^{-(r-5)^2}$  is quite different from the asymptotic profile  $\psi_{Q_B}$ , indicating that the standing-ring  $\psi_{Q_B}$  profile (25) is an attractor in the radial case.

### 3 Shrinking-ring solutions of the supercritical BNLS ( $4/d < \sigma < 4$ )

In this section, we consider the regime  $4/d < \sigma < 4$ . In the NLS analogue ( $2/d < \sigma < 2$ ), the asymptotic profile has a “double-lens” radial-phase term, see (5a), whose explicit form is used in the asymptotic calculation of the blowup rate and shrinking rate. In contrast, for the BNLS we do not know the corresponding “double-lens” radial phase term, but only the amplitude  $|\psi_{Q_B}|$ . Therefore, the results of the asymptotic analysis are weaker, and we need to rely on the analogy between the NLS and the BNLS.

#### 3.1 Informal analysis

We consider singular ring-type solutions of the supercritical BNLS with  $4/d < \sigma < 4$ , that undergo a quasi self-similar collapse with the asymptotic profile  $\psi_{Q_B}$ , whose amplitude is given by

$$|\psi_{Q_B}(t, r)| = \frac{1}{L^{2/\sigma}(t)} |Q_B(\rho)|, \quad \rho = \frac{r - r_{\max}(t)}{L}, \quad r_{\max}(t) \sim r_0 L^\alpha(t). \quad (27)$$

As before, we assume that  $\psi \sim \psi_{Q_B}$  in the region  $r_{\max} - r = \mathcal{O}(L)$ , i.e., for  $|\rho| \leq \rho_c = \mathcal{O}(1)$ . We assume that  $\alpha \leq 1$ , since otherwise the rings are unstable. Indeed, if  $\alpha > 1$ , then  $\rho = r/L + o(1)$ , and the rings eventually evolve into a peak solution.

We first derive a lower bound for  $\alpha$ :

**Lemma 5.** *Let  $4/d < \sigma < 4$ , and let  $\psi$  be a ring-type singular solution of the BNLS equation (10), whose asymptotic profile is of the form (27) with  $\alpha \leq 1$ . Then,*

$$\alpha \geq \alpha_B,$$

where

$$\alpha_B = \frac{4 - \sigma}{\sigma(d - 1)} > 0.$$

Therefore, the ring is shrinking, i.e.,  $\lim_{t \rightarrow T_c} r_{\max}(t) = 0$ .

*Proof.* First, since  $4/d < \sigma < 4$ , then  $0 < \alpha_B < 1$ . The power of the collapsing core  $\psi_{Q_B}$  is

$$\begin{aligned}\|\psi_{Q_B}\|_2^2 &= L^{-4/\sigma} \int_{r=r_{\max}-\rho_c \cdot L(t)}^{r_{\max}+\rho_c \cdot L(t)} \left| Q_B \left( \frac{r-r_{\max}}{L} \right) \right|^2 r^{d-1} dr \\ &= L^{-4/\sigma} \int_{\rho=-\rho_c}^{\rho_c} |Q_B(\rho)|^2 (L\rho + r_0 L^\alpha)^{d-1} (L d\rho).\end{aligned}$$

In the case  $\alpha < 1$ , we have that  $L|\rho| \leq L\rho_c \ll r_0 L^\alpha$ , hence  $L\rho + r_0 L^\alpha \sim r_0 L^\alpha$ . Therefore,

$$\|\psi_{Q_B}\|_2^2 \sim L^{1-4/\sigma+\alpha(d-1)}(t) \cdot r_0^{\alpha(d-1)} \int_{\rho=-\rho_c}^{\rho_c} |Q_B(\rho)|^2 d\rho.$$

In the case  $\alpha = 1$ , we have that  $L\rho + r_0 L^\alpha = (r_0 + \rho)L$ , hence

$$\|\psi_{Q_B}\|_2^2 \sim L^{1-4/\sigma+\alpha(d-1)}(t) \int_{\rho=-\rho_c}^{\rho_c} |Q_B(\rho)|^2 (r_0 + \rho)^{d-1} d\rho.$$

In both cases  $\|\psi_{Q_B}\|_2^2 = \mathcal{O}(L^{1-4/\sigma+\alpha(d-1)})$ . Since  $\|\psi_{Q_B}\|_2^2 \leq \|\psi\|_2^2 = \|\psi_0\|_2^2 < \infty$ , then  $L^{1-4/\sigma+\alpha(d-1)}$  has to be bounded as  $L \rightarrow 0$ . Therefore,  $1 - 4/\sigma + \alpha(d-1) \geq 0$ , from which the result follows.  $\square$

Let

$$P_{\text{collapse}} = \liminf_{\varepsilon \rightarrow 0^+} \lim_{t \rightarrow T_c} \int_{r < \varepsilon} |\psi|^2 r^{d-1} dr$$

be the amount of power that collapses into the singularity. We say that the solution  $\psi$  undergoes a *strong collapse* if  $P_{\text{collapse}} > 0$ , and a *weak collapse* if  $P_{\text{collapse}} = 0$ .

**Corollary 6.** *Under the conditions of Lemma 5,  $\psi_{Q_B}$  undergoes a strong collapse if  $\alpha = \alpha_B$ , and a weak collapse if  $\alpha > \alpha_B$ .*

*Proof.* This follows directly from the proof of Lemma 5.  $\square$

In the NLS with  $2/d < \sigma < 2$ , the shrinking rings undergo a strong collapse with  $\alpha = \alpha_{\text{NLS}}$ , see (5c). Therefore, by analogy, we expect that the shrinking rings of the BNLS will also undergo a strong collapse, in which case  $\alpha = \alpha_B$ .

The blowup rate of singular shrinking-ring solutions of the NLS with  $2/d < \sigma < 2$  is [FGW07]

$$p = \frac{1}{1 + \alpha_{\text{NLS}}} = \frac{1}{2 - \frac{\sigma d - 2}{\sigma(d-1)}}.$$

From the analogy of the BNLS with the NLS (up to the change  $2 \rightarrow 4$ ), we expect that the blowup rate of singular shrinking rings of the BNLS is

$$p = \frac{1}{4 - \frac{\sigma d - 4}{\sigma(d-1)}} = \frac{1}{3 + \alpha_B}.$$

Therefore, we have the following Conjecture:

**Conjecture 7.** *Let  $d > 1$  and  $4/d < \sigma < 4$ , and let  $\psi$  be a singular ring-type solution of the BNLS (10). Then,*

1. *The solution is quasi self-similar, i.e.,  $\psi \sim \psi_{Q_B}$  for  $r - r_{\max} = \mathcal{O}(L)$ , where  $\psi_{Q_B}$  is given by (27).*

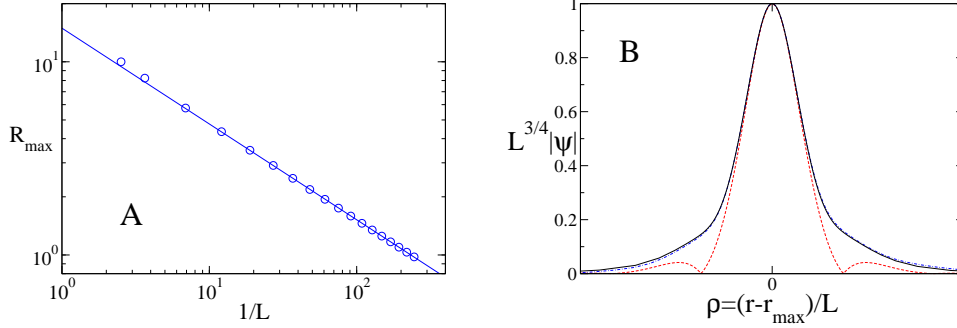


Figure 7: Ring-type singular solution of the supercritical BNLS (10) with  $d = 2$  and  $\sigma = 8/3$ . A)  $r_{\max}$  as a function of the focusing factor  $1/L$ . Solid line is  $r_{\max} = 14.9L^{0.496}$ . B) The rescaled solution, see (26), at  $L(t) = 10^{-1}$  (blue dash-dotted line) and  $L(t) = 10^{-2}$  (black solid line). Red dashed line is the rescaled one-dimensional ground state  $|R_{B,1D}(x)|$ .

2. The solution is a shrinking ring, i.e.,  $\lim_{t \rightarrow T_c} r_{\max}(t) = 0$ .

3. The shrinking rate is

$$\alpha = \alpha_B = \frac{4 - \sigma}{\sigma(d - 1)}. \quad (28)$$

Specifically,  $0 < \alpha < 1$ .

4. The blowup rate is

$$L(t) \sim \kappa(T_c - t)^p, \quad p = \frac{1}{3 + \alpha_B} = \frac{1}{4 - \frac{\sigma d - 4}{\sigma(d - 1)}}. \quad (29)$$

Specifically,  $\frac{1}{4} < p < \frac{1}{3}$ .

In Section 3.2 we provide numerical evidence in support of Conjecture 7.

### 3.2 Simulations

The supercritical BNLS equation with  $d = 2$  and  $\sigma = 8/3$  was solved with the initial condition  $\psi_0 = 2e^{(r-10)^2}$ . The simulation was run up to a focusing level of  $L(t) = 10^4$ .

We next test Conjecture 7 numerically, clause by clause.

- Figure 7A shows that the ring shrinks at a rate of  $r_{\max}(t) \sim 14.9L^\alpha(t)$  with  $\alpha \approx 0.496$ , which is close to the predicted value of

$$\alpha_B = \frac{4 - \frac{8}{3}}{\frac{8}{3}(2 - 1)} = \frac{1}{2}.$$

- In Figure 7B we plot the solution, rescaled according to (26), at the focusing levels  $1/L = 10$  and  $1/L = 100$ . The two curves are indistinguishable for  $\rho = \mathcal{O}(1)$ , but not for all  $\rho$ , showing that the solution undergoes a quasi self-similar collapse with the  $\psi_{Q_B}$  profile (27).

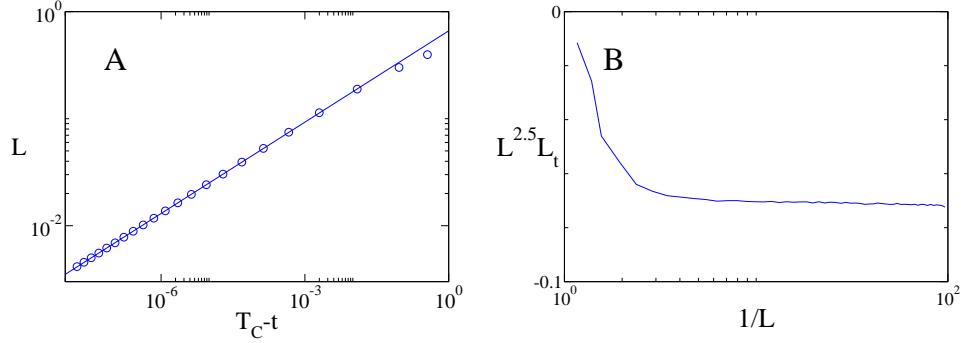


Figure 8: Blowup rate of the solution of Figure 7. A) Solution width  $L$  as a function of  $(T_c - t)$  on a logarithmic scale (circles). Solid line is  $L = 0.662(T_c - t)^{0.2844}$ . B)  $L^{2.5} L_t$  as a function of  $1/L$ .

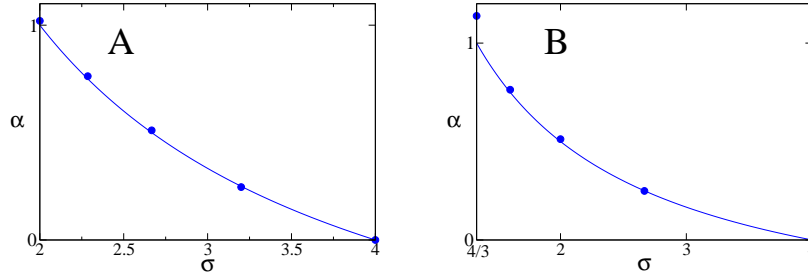


Figure 9: The numerical shrinking rate  $\alpha$  (circles) for ring-type singular solutions of the BNLS with  $4/d \leq \sigma \leq 4$ . The solid line is  $\alpha = \alpha_B(\sigma, d)$ , see (28). A)  $d = 2$  and  $\sigma = 2, 16/7, 8/3, 16/5$  and 4. B)  $d = 3$  and  $\sigma = 4/3, 8/5, 2, 8/3$  and 4.

3. Figure 7B shows that the self-similar profile of the standing-ring solution is close to  $R_{B,1D}(\xi)$ , the one-dimensional ground-state of equation (17b).
4. Figure 8A shows that  $L(t) \sim 0.662(T_c - t)^{0.2844}$ . Therefore, the calculated blowup rate  $p = 0.282$  is close to the predicted value of  $p = \frac{1}{3+\alpha_B} = 1/3.5 \approx 0.2857$ .
5. In order to check whether the blowup rate of  $L$  is exactly  $p = 1/3.5$ , we compute the limit  $\lim_{t \rightarrow T_c} L^{2.5} L_t$ . Recall that if  $L(t) \sim \kappa(T_c - t)^{1/3.5}$ , then

$$\lim_{t \rightarrow T_c} L^{2.5} L_t = -\frac{\kappa^{3.5}}{3.5} < 0,$$

while for a faster blowup rate,  $L^{2.5} L_t \rightarrow 0$ , and for a slower blowup rate,  $L^{2.5} L_t \rightarrow -\infty$ . Figure 8B shows that  $L^{2.5} L_t$  converges to a negative constant, implying that the blowup rate is exactly  $p = 1/3.5$ .

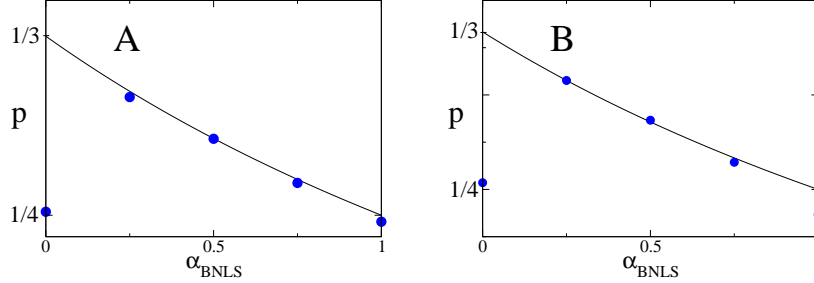


Figure 10: Same as Figure 9, for the numerical blowup rate  $p$ , defined by  $L(t) \sim c(T_c - t)^p$ , as a function of  $\alpha_B$ . Solid line is  $p = \frac{1}{3 + \alpha_B}$ . The calculated values of  $p$  for  $1/2 \leq \alpha_B \leq 1$  are slightly lower than the predicted value  $\frac{1}{3 + \alpha_B}$ .

In Figure 9 we present the numerical values of the shrinking parameter  $\alpha$ , defined by  $r_{\max}(t) \sim cL^\alpha$  for ten different values of  $(\sigma, d)$ . In all cases, the value of  $\alpha$  is very close to  $\alpha_B$ , see (28). In Figure 10 we present the numerical values of the blowup rate  $p$  for the same simulations, and find that they are close to  $p = 1/(3 + \alpha_B)$  for  $\alpha_B > 0$ . At  $\alpha_B = 0+$ ,  $p$  has a jump discontinuity to  $1/4$ , in accordance with Conjecture 4. The discontinuity at  $\alpha_B = 0$  ( $\sigma = 4$ ) is a manifestation of the phase transition from shrinking to standing rings, see Section 1.4.

## 4 Equal-rate shrinking rings (critical BNLS)

### 4.1 Informal Analysis

We consider singular ring-type solutions of the critical BNLS, that undergo a quasi self-similar collapse with the asymptotic profile

$$\psi_{Q_B}(t, r) = \frac{1}{L^{d/2}(t)} Q_B \left( \frac{r - r_{\max}(t)}{L(t)} \right) e^{i \int^t \frac{1}{L^4(s)} ds}, \quad r_{\max} \sim r_0 L^\alpha. \quad (30)$$

**Lemma 8.** *Let  $\psi_{Q_B}(t, r)$ , see (30), be the asymptotic profile of singular ring-type solutions of the critical BNLS (12). Then,  $\alpha = 1$ .*

*Proof.* Theorem 3 implies that the collapsing core of singular solutions of the critical BNLS is self-similar in  $r/L$ , i.e.,

$$|\psi(t, r)| \sim \frac{1}{L^{d/2}} \left| \Psi \left( \frac{r}{L} \right) \right|.$$

Therefore, a singular solution of the critical BNLS is ring-type if and only if  $|\Psi(\rho)|$  attains its maximum at some  $\rho_{\max} > 0$ . Hence,  $r_{\max}(t) = \rho_{\max} \cdot L(t)$ . Therefore,  $\alpha = 1$ .  $\square$

By Theorem 2, the blowup rate is lower-bounded by  $1/4$ . We recall that ring-type singular solutions of the critical NLS have a square-root blowup rate [FGW05]. Therefore, we expect that ring-type singular solutions of the critical BNLS have a quartic-root blowup rate.

In summary, we conjecture the following:

**Conjecture 9.** *Let  $\psi$  be ring-type singular solution of the critical BNLS (12). Then,*



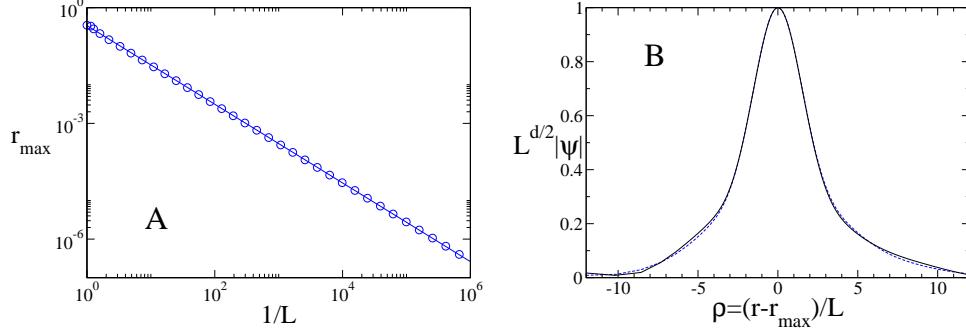


Figure 11: Ring-type singular solution of the critical BNLS (12) with  $d = 2$ . A)  $r_{\max}$  as a function of  $1/L$ . Solid line is  $r_{\max} = 79.5L^{1.02}$ . B) The rescaled solution, see (26), at  $L(t) = 10^{-3}$  (blue dashed line) and  $L(t) = 10^{-6}$  (black solid line).

1. The solution undergoes an equal-rate collapse, i.e.,  $r_{\max}(t) \sim r_0 L(t)$ .
2. The solution undergoes a quasi self-similar collapse with the asymptotic profile

$$\psi_{Q_B}(t, r) = \frac{1}{L^{d/2}(t)} Q_B \left( \frac{r - r_0 L}{L} \right) e^{i \int_{s=0}^t \frac{1}{L^4(s)} ds}. \quad (31a)$$

3. The blowup rate is exactly a quartic root, i.e.,

$$L(t) \sim \kappa \sqrt[4]{T_c - t}, \quad \kappa > 0. \quad (31b)$$

## 4.2 Simulations

The critical BNLS (12) with  $d = 2$  was solved with the initial condition  $\psi_0 = 2.5e^{(r-10)^2}$ . The simulation was run up to a focusing level of  $L(t) = 10^{-6}$ . We next test Conjecture 9 numerically, clause by clause.

1. Figure 11A shows that the ring shrinks at a rate of  $r_{\max}(t) \sim cL^\alpha(t)$  with  $\alpha \approx 1.02$ , which is close to the predicted value of  $\alpha = 1$ .
2. In Figure 11B we plot the solution, rescaled according to (26), at the focusing levels  $1/L = 10^3$  and  $1/L = 10^6$ . The two curves are indistinguishable, showing that the solution undergoes a quasi self-similar collapse with the  $\psi_{Q_B}$  profile (31a).
3. Figure 12A shows that  $L(t) \sim 0.433(T_c - t)^{0.2477}$ . Therefore, the calculated blowup rate is close to a quartic root.
4. By Conjecture 9, the blowup rate of  $L(t)$  should be exactly  $1/4$ , hence  $L^3 L(t)$  should converge to a negative constant. However, in Figure 12B,  $L^3 L_t$  does not converge to a constant, but rather slowly decreases away from zero. This indicates that the blowup rate is slower than a quartic-root, which is in contradiction with Theorem 2. There are two possible explanations for this:
  - (a) It may be that the numerical finding that  $\alpha$  is slightly above 1 and  $p$  is slightly below  $1/4$  is an artifact of our numerical method, see Section 5. Indeed, in all our simulations for  $1/2 \leq \alpha < 1$  in

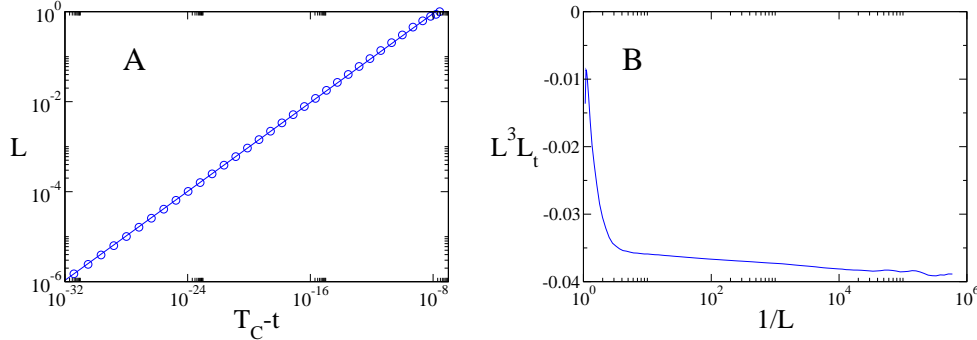


Figure 12: Blowup rate of the solution of Figure 11. A) Solution width  $L$  as a function of  $(T_c - t)$  on a logarithmic scale (circles). Solid line is  $L = 0.433(T_c - t)^{0.2476}$ . B)  $L_t L^3$  as a function of  $1/L$ .

Figures 9 and 10, the calculated values of the shrinking rate  $\alpha$  were all slightly above  $\alpha_B$ , and the blowup rates were slightly below  $\frac{1}{3+\alpha_B}$ . In those cases, however, these small differences did not change the qualitative behavior of the solution. In contrast, a small increment (whether numerical or genuine) from  $\alpha = 1$  will drastically change the dynamics, from equal-rate ring-type solutions into peak-type solutions.

- (b) It may be that ring-type solutions of the critical BNLS are only meta-stable, having shrinking rates  $\alpha > 1$  and a blowup rate slower than  $1/4$ . This does not contradict with Theorem 2, since in this case the ring-type solutions will eventually evolve into peak-type solutions with a different blowup rate.

We do not know which of the two options is true.

## 5 Numerical Method: Adaptive mesh construction

In this study, we computed singular solutions of the BNLS equation (10). These solutions become highly-localized, so that the spatial scale-difference between the singular region  $r - r_{\max} = \mathcal{O}(L)$  and the exterior regions can be as large as  $\mathcal{O}(1/L) \sim 10^{10}$ . In order to resolve the solution at both the singular and non-singular regions, we use an adaptive grid.

We generate the adaptive grids using the *Static Grid Redistribution* (SGR) method, which was first introduced by Ren and Wang [RW00], and later simplified and improved by Gavish and Ditzkowsky [DG09]. Using this approach, the solution is allowed to propagate (self-focus) until it becomes under-resolved. At this stage, a new grid, with the same number of grid-points, is generated using De'Boors 'equidistribution principle', wherein the grid points  $\{r_m\}$  are spaced such that a certain weight function  $w_1[\psi]$  is equidistributed, i.e., that

$$\int_{r=r_m}^{r_{m+1}} w_1[\psi(r)] dr = \text{const},$$

see [RW00, DG09] for details.

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**Algorithm 1** The SGR method, as implemented in [DG09].

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1. Find a nonlinear coordinate transformation  $r(x) : [0, 1] \rightarrow [0, R]$ , under which the weight function  $w[\psi(r(x))]$  becomes uniformly distributed.
2. Transform the solution and equation to the new coordinate system. For example, the second spatial derivative in the NLS transforms as

$$\psi_{rr} \mapsto \psi_{xx} r_x^2 + \psi_x r_{xx}.$$

3. Approximate the equation on a uniform grid  $\{x_m\}$ , using standard finite-differences (or another method of choice).
- 

The method of [DG09] is given in Algorithm 1. Note that, since  $r(x)$  is nonlinear, the mapping of the derivatives of  $\psi$  (step 2) involves nonlinear combinations of the derivatives of  $r$ . This is not a great problem for the NLS, which has only second-order derivatives, but becomes much messier for the biharmonic operator (11), with its many high-order derivatives of  $\psi$ .

Therefore, in this study we implement a simplified version of the method of [DG09], which is given in Algorithm 2, which uses a non-uniform grid in the old-coordinate system, and thereby dispenses with the need for transforming *the equation*, and is much easier to implement in the biharmonic case.

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**Algorithm 2** The SGR method, as implemented in this work.

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1. Find a nonlinear coordinate transformation  $r(x) : [0, 1] \rightarrow [0, R]$ , under which the weight function  $w[\psi(r(x))]$  becomes uniformly distributed.
  2. Create the uniform grid in the transformed system  $\{x_m\}$ .
  3. Create the (highly) non-uniform grid  $r_m = r(x_m)$  in the original (physical) coordinate system.
  4. On the non-uniform grid, approximate the equation using standard (non-uniform) finite-differences.
- 

We use a third-order accurate finite-difference approximation of the radial biharmonic operator (11), with a seven-point stencil.

The method in [DG09] allows control of the fraction of grid points that migrate into the singular region, preventing under-resolution at the exterior regions. This is done by using a weight-function  $w_2$ , which penalizes large inter-grid distances. However, we found that this numerical mechanism, while necessary, is insufficient for our purposes. In order to understand the reason, let us consider the grid-point spacings  $\Delta r_m = r_{m+1} - r_m$ . Using the method of [DG09] with both  $w_1$  and  $w_2$  causes a very sharp bi-partition of the grid points – to those inside the singular region, whose spacing is determined by  $w_1$  and is  $\Delta r_m = \mathcal{O}(L)$ , and to those outside the singular region, whose spacing is determined by  $w_2$  and is  $\Delta r_m = \mathcal{O}(1)$ , see Figure 13A. Inside each of these regions, the finite difference approximation we use is well conditioned. However, at the transition between these two regions, the finite-difference stencil, seven-points in width, spans grid-spacings with  $\mathcal{O}(1/L)$  scale-difference — leading to under-resolution which completely violates the validity of the finite-difference approximation.

In order to overcome this limitation, we improve the algorithm of [DG09] by adding a third weight

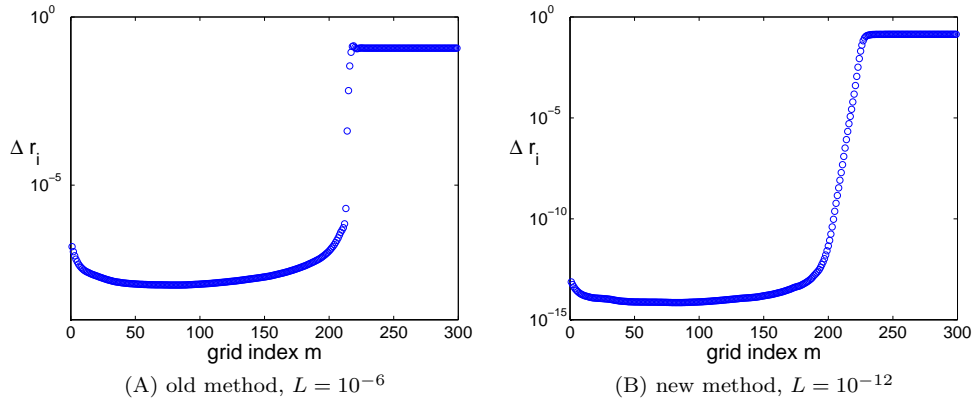


Figure 13: The grid-spacing  $\Delta r_m$  obtained using the SGR method of [DG09] for a peak-type singular solution of the BNLS. A) The grid generated the original method of [DG09] at focusing level of  $L = 10^{-6}$ . The Singular and non-singular regions are well-resolved, but the transition region  $\Delta r_m$  displays a discontinuity. At this point, the finite difference operator becomes ill-conditioned. B) same as (A), after adding the new penalty function  $w_3$ , at focusing level  $L = 10^{-12}$ . Even at this much larger focusing level, the transition region is now well resolved.

function

$$w_3(r_m) = \sqrt{1 + \frac{|\Delta^2 r_m|}{\Delta r_m}},$$

which penalizes the second-difference  $\Delta^2 r_m = \Delta r_{m+1} - \Delta r_m$  operator of the grid locations, allowing for a smooth transition between the singular region and the non-singular region, see Fig 13B.

On the sequence of grids, the equations are solved using a Predictor-Corrector Crank-Nicholson scheme, which is second-order in time.

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